STRING AND DILATON EQUATIONS FOR COUNTING LATTICE POINTS IN THE MODULI SPACE OF CURVES.

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Abstract. We apply the techniques of Eynard and Orantin to the plane curve $xy - y^2 = 1$ to enumerate covers of the two-sphere branched over three points. This produces new recursion relations—string and dilaton equations—between the piecewise polynomials that enumerate such covers.

1. Introduction

Consider genus g branched covers of S^2 branched over 0, 1 and ∞ with ramification $(b_1, ..., b_n)$ over ∞, ramification $(2, 2, ..., 2)$ over 1 and ramification greater than 1 at all points above 0. The number of connected topologically inequivalent such coverings $N_{g,n}(b_1, ..., b_n)$ was studied in [11] where it was shown that there exists polynomials $N^{(k)}_{g,n}(b_1, ..., b_n)$ for $k = 1, ..., n$ such that $N_{g,n}(b_1, ..., b_n)$ decomposes

$$N_{g,n}(b_1, ..., b_n) = N^{(k)}_{g,n}(b_1, ..., b_n), \quad k = \text{number of odd } b_i.$$

In fact only even k is necessary since by definition $N_{g,n}(b_1, ..., b_n)$ vanishes if the number of odd b_i is odd. Each $N^{(k)}_{g,n}(b_1, ..., b_n)$ is a polynomial in the b_i^2, symmetric under permutations that preserve the parity of the b_i. The number $N_{g,n}(b_1, ..., b_n)$ is presented in [11] in terms of counting lattice points inside integral convex polytopes depending on $(b_1, ..., b_n)$ which make up a cell decomposition of $\mathcal{M}_{g,n}$, the moduli space of genus g curves with n labeled points, and hence is said to count lattice points in the moduli space of curves.

Eynard and Orantin [3] have developed a sequence of invariants of plane curves to study enumerative problems in geometry. For a curve $C \subset \mathbb{C}^2$ the invariants are multilinear differentials, which are meromorphic differentials $\omega^{(g)}_n(z_1, ..., z_n)dz_1...dz_n$ on $C \times C \times ... \times C$, that satisfy recursion relations with a Virasoro algebra structure. See Section 2.2 for more details.

The generating function

$$F^{(g)}_n(z_1, ..., z_n) = \sum_{b_i > 0} N_{g,n}(b_1, ..., b_n) z_1^{b_1} ... z_n^{b_n}$$

has radius of convergence of 1 in each variable, and extends to a meromorphic function in each variable on the whole complex plane. See Lemma 2 in Section 3.

It was proven in [11] that the $N_{g,n}(b_1, ..., b_n)$ satisfy recursion relations which uniquely determine them from $N_{0,3}(b_1, b_2, b_3) = 1$ (when $b_1 + b_2 + b_3$ is even and zero otherwise.) These recursion relations are used to prove the following theorem.

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Theorem 1. *For* $2g-2+n>0$
$$\omega_n^{(g)}=\frac{\partial}{\partial z_1}...\frac{\partial}{\partial z_n}F_n^{(g)}dz_1...dz_n$$
are the Eynard-Orantin invariants of the plane curve $xy-y^2=1$.

The Eynard-Orantin invariants satisfy further recursion relations known as string and dilaton equations. See Section 4. They give rise to new recursion relations between the $N_{g,n}$. The first two of these are a consequence of the string equations.

Theorem 2.
$$N_{g,n+1}(1,b_1,...,b_n)=\sum_{j=1}^{n}\sum_{k=1}^{b_j}kN_{g,n}(b_1,...,b_n)|_{b_j=k}$$
$$N_{g,n+1}(2,b_1,...,b_n)=\sum_{j=1}^{n}\sum_{k=1}^{b_j}kN_{g,n}(b_1,...,b_n)|_{b_j=k}-\frac{1}{2}\sum_{j=1}^{n}b_jN_{g,n}(b_1,...,b_n)$$

Corollary 1. *The string equations determine the genus 0 invariants.*

The count of branched covers $N_{g,n}(b_1,...,b_n)$ requires the b_i to be positive integers. Nevertheless the polynomials $N_{g,n}^{(k)}(b_1,...,b_n)$ can be evaluated at $b_i=0$ so we use them to define $N_{g,n}(b_1,...,b_n)$ when some of the $b_i=0$. For example,

$$N_{g,n}(0,0,...,0):=N_{g,n}^{(0)}(0,0,...,0)=\chi(\mathcal{M}_{g,n}) \tag{1}$$

for $\chi(\mathcal{M}_{g,n})$ the (orbifold) Euler characteristic of $\mathcal{M}_{g,n}$ was proven in [11]. A consequence of the dilaton equation is the following new recursion relation between the $N_{g,n}$.

Theorem 3.
$$N_{g,n+1}(2,b_1,...,b_n)-N_{g,n+1}(0,b_1,...,b_n)=(2g-2+n)N_{g,n}(b_1,...,b_n)$$

Using the dilaton equation, Eynard and Orantin show how to essentially allow $n=0$ in their meromorphic form invariants. This enables them to define symplectic invariants $F^{(g)}(C)=\omega_0^{(g)}$ for $g>1$ of a plane curve $C\subset\mathbb{C}^2$ which are invariant under symplectic transformations of $\mathbb{C}^2$.

Corollary 2. *The Eynard-Orantin symplectic invariants of the plane curve* $xy-y^2=1$ *are the orbifold Euler characteristics of the classical moduli space.*
$$F^{(g)}=\frac{N_{g,1}(0)}{2-2g}=\chi(\mathcal{M}_g).$$

In Section 2 we give a short introduction to $N_{g,n}(b_1,...,b_n)$ and $\omega_n^{(g)}(p_1,...,p_n)$. Section 3 contains the proof of Theorem 1 and Section 4 contains the proofs of Theorems 2 and 3. In Section 4.2 we briefly describe string and dilaton equations for the Weil-Petersson volumes of the moduli spaces of hyperbolic surfaces $\mathcal{M}_{g,n}(\mathbf{L})$, which were proven by Mirzakhani [9] to be polynomial in the lengths $L_1,...,L_n$ of the geodesic boundary components, and point out their similarity to the work here. Section 5 contains vanishing results for $N_{g,n}(b_1,...,b_n)$ when some of the b_i are zero and hence it does not arise from a counting problem. Examples are given in Section 6.

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2. Background

In this section we give a short introduction to the two main ingredients of the paper—the piecewise polynomials $N_{g,n}(b_1,...,b_n)$, and the Eynard-Orantin invariants $\omega_n^{(g)}(p_1,...,p_n)$.

2.1. Lattice count polynomials.

Let $\mathcal{M}_{g,n}$ be the moduli space of genus g curves with n labeled points. The *decorated* moduli space $\mathcal{M}_{g,n}\times\mathbb{R}_+^n$ equips the labeled points with positive numbers $(b_1,...,b_n)$ [12]. It has a cell decomposition due to Penner, Harer, Mumford and Thurston

$$\mathcal{M}_{g,n}\times\mathbb{R}_+^n\cong\bigcup_{\Gamma\in\mathcal{F}\mathrm{at}_{g,n}}P_\Gamma \tag{2}$$

where the indexing set $\mathcal{F}\mathrm{at}_{g,n}$ is the space of labeled fatgraphs of genus g and n boundary components. A *fatgraph* is a graph Γ with vertices of valency >2 equipped with a cyclic ordering of edges at each vertex. The cell decomposition (2) arises by the existence and uniqueness of meromorphic quadratic differentials with foliations having compact leaves, known as Strebel differentials which can be described via labeled fatgraphs with lengths on edges. Restricting this homeomorphism to a fixed n-tuple of positive numbers $(b_1,...,b_n)$ yields a space homeomorphic to $\mathcal{M}_{g,n}$ decomposed into compact convex polytopes

$$P_\Gamma(b_1,...,b_n)=\{\mathbf{x}\in\mathbb{R}_+^{E(\Gamma)}|A_\Gamma\mathbf{x}=\mathbf{b}\}$$

where $\mathbf{b}=(b_1,...,b_n)$ and $A_\Gamma:\mathbb{R}^{E(\Gamma)}\to\mathbb{R}^n$ is the incidence matrix that maps an edge to the sum of its two incident boundary components.

When the b_i are positive integers the polytope $P_\Gamma(b_1,...,b_n)$ is an integral polytope and we define $N_\Gamma(b_1,...,b_n)$ to be its number of positive integer points. The weighted sum of N_Γ over all labeled fatgraphs of genus g and n boundary components is the piecewise polynomial [11]

$$N_{g,n}(b_1,...,b_n)=\sum_{\Gamma\in\mathcal{F}\mathrm{at}_{g,n}}\frac{1}{|Aut\Gamma|}N_\Gamma(b_1,...,b_n)$$

The top homogeneous degree terms of the polynomials $N_{g,n}^{(k)}$ (k even) representing $N_{g,n}$ coincides with Kontsevich's volume polynomial [8]

$$V_{g,n}(b_1,...,b_n)=\sum_{\Gamma\in\mathcal{F}\mathrm{at}_{g,n}}\frac{1}{|Aut\Gamma|}V_\Gamma(b_1,...,b_n)$$

where $V_\Gamma(b_1,...,b_n)$ is the volume of $P_\Gamma(b_1,...,b_n)$ induced from the Euclidean volumes on $\mathbb{R}^{E(\Gamma)}$ and $\mathbb{R}^n$.

Each integral point in the polytope $P_\Gamma(b_1,...,b_n)$ corresponds to a Dessin d'enfants defined by Grothendieck [6] which is a branched cover of S^2 branched over 0, 1 and ∞ with ramification $(b_1,...,b_n)$ over ∞, ramification $(2,2,...,2)$ over 1 and ramification greater than 1 at all points above 0. The branched cover viewpoint gives an easier definition of $N_{g,n}(b_1,...,b_n)$ while the description via lattice points in $\mathcal{M}_{g,n}$ leads to the relation of $N_{g,n}(b_1,...,b_n)$ with the Euler characteristic of $\mathcal{M}_{g,n}$ given in (1) and the relation of its top coefficients with intersection numbers on $\overline{\mathcal{M}}_{g,n}$, [11].

2.2. **Eynard-Orantin invariants.** Eynard and Orantin [3] associate multilinear differentials to any Riemann surface C equipped with two meromorphic functions x and y with the property that the branch points of x are simple and the map

$$\begin{array}{ccc} C & \to & \mathbb{C}^2 \\ p & \mapsto & (x(p), y(p)) \end{array}$$

is an immersion.

If $f(z)$ is a meromorphic function on $\overline{\mathbb{C}}$ and analytic at z_0 then $f(z_0)$ can be expressed in terms of local information around the poles of f as follows:

$$f(z_0) = \operatorname*{Res}_{z=z_0} \frac{f(z)dz}{z-z_0} = -\sum_{\alpha} \operatorname*{Res}_{z=\alpha} \frac{f(z)dz}{z-z_0}$$

where the sum is over all poles α of $f(z)$. Similarly, a meromorphic 1-form on $\overline{\mathbb{C}}$ can be expressed in terms of local information around its poles

$$\omega(z_0) = f(z_0)dz_0 = \operatorname*{Res}_{z=z_0} \frac{f(z)dz}{z-z_0}dz_0 = \operatorname*{Res}_{z=z_0} \frac{dz_0}{z-z_0}f(z)dz \tag{3}$$

$$= \sum_{\alpha} \operatorname*{Res}_{z=\alpha} \frac{dz_0}{z_0-z}\omega(z)$$

where the sum is over all poles α of $\omega(z)$.

The expression $\dfrac{dz_0}{z_0-z}\omega(z)$ in (3) is an example of a *bilinear differential* which is a meromorphic differential on $\overline{\mathbb{C}} \times \overline{\mathbb{C}}$. Another basic example of a bilinear differential on a Riemann surface C of any genus arises from the meromorphic differential $\eta_w(z)dz$ unique up to scale which has a double pole at $w \in C$ and all A-periods vanishing. The scale factor can be chosen so that $\eta_w(z)dz$ varies holomorphically in w, and transforms as a 1-form in w and hence it is naturally expressed as the unique bilinear differential on C

$$B(w,z) = \eta_w(z)dwdz, \quad \oint_{A_i} B = 0, \quad B(w,z) \sim \frac{dwdz}{(w-z)^2} \text{ near } w = z.$$

It is symmetric in w and z. We will call $B(w,z)$ the *Bergmann Kernel*, following [3]. It is called the fundamental normalised differential of the second kind on C in [5]. Recall that a differential is *normalised* if its A-periods vanish and it is of the *second kind* if its residues vanish.

For every $(g,n) \in \mathbb{Z}^2$ with $g \geq 0$ and $n > 0$ Eynard and Orantin [3] define a multilinear differential, i.e. a tensor product of meromorphic 1-forms on the product C^n, notated by $\omega_n^{(g)}(p_1, ..., p_n)$ for $p_i \in C$. When $2g-2+n > 0$, $\omega_n^{(g)}(p_1, ..., p_n)$ is defined recursively in terms of local information around the poles of $\omega_{n'}^{(g')}(p_1, ..., p_n)$ for $2g'+2-n' < 2g-2+n$. This is closely related to (3) and its generalisation to any Riemann surface C which expresses a normalised differential of the second kind in terms of local information around its poles using $\eta_{z_0,p}(z) = \int_p^{z_0} B(z,z')$ on C in place of the Cauchy kernel $dz_0/(z_0-z)$.

For $2g-2+n > 0$, the poles of $\omega_n^{(g)}(p_1, ..., p_n)$ occur at the branch points of x. Since each branch point α of x is simple, for any point $p \in C$ close to α there is a unique point $\bar{p} \neq p$ close to α such that $x(\bar{p}) = x(p)$. The recursive definition

of $\omega_n^{(g)}(p_1, ..., p_n)$ uses only local information around branch points of x and makes use of the well-defined map $p \mapsto \bar{p}$ there.

Set $\omega_1^{(0)} = 0$ (which agrees with the convention in [3] but disagrees with the convention in [2].)

$$\omega_2^{(0)} = B(w, z) \tag{4}$$

For $2g - 2 + n > 0$,

(5)

$$\omega_{n+1}^{(g)}(z_0, z_S) = \sum_\alpha \operatorname*{Res}_{z=\alpha} K(z_0, z)\Bigg[\omega_{n+2}^{(g-1)}(z, \bar{z}, z_S) + \sum_{\substack{g_1+g_2=g \\ I \sqcup J = S}} \omega_{|I|+1}^{(g_1)}(z, z_I)\omega_{|J|+1}^{(g_2)}(\bar{z}, z_J)\Bigg]$$

where the sum is over branch points α of x, $S = \{1, ..., n\}$ and

$$K(z_0, z) = \frac{-\int_{\bar{z}}^{z} B(z_0, z')}{2(y(z) - y(\bar{z}))dx(z)}$$

is well-defined in the vicinity of each branch point of x. Note that the quotient of a differential by the differential $dx(z)$ is a meromorphic function.

The recursion (5) depends only on the meromorphic differential ydx and the map $p \mapsto \bar{p}$ around branch points of x. The simplest example of a plane curve with non-trivial Eynard-Orantin invariants is $y^2 = x$. It is known as the Airy curve since the Eynard-Orantin invariants reproduce Kontsevich's generating function [8] for intersection numbers on the moduli space.

The simplicity of the curve $y^2 = x$ can be measured by the divisor of its differential ydx which is $(ydx) = 2(0) - 4(\infty)$. The plane curve $xy - y^2 = 1$ also has extremely simple divisor $(ydx) = (-1) + (1) - (0) - 3(\infty)$. The branch points of x are ± 1, and the map $p \mapsto \bar{p} = 1/p$ is global. In this case $x(z) = z + 1/z$, $y(z) = z$ and (5) becomes

$$\omega_{n+1}^{(g)}(z_0, z_S) = \sum_{\alpha=\pm 1} \operatorname*{Res}_{z=\alpha} K(z_0, z)\Bigg[\omega_{n+2}^{(g-1)}(z, 1/z, z_S) + \sum_{\substack{g_1+g_2=g \\ I \sqcup J = S}} \omega_{|I|+1}^{(g_1)}(z, z_I)\omega_{|J|+1}^{(g_2)}(1/z, z_J)\Bigg]$$

3. Recursion

In [11] the piecewise polynomials $N_{g,n}(b_1, ..., b_n)$ were shown to satisfy the following recursion which uniquely determines $N_{g,n}$ from $N_{0,3}$ and $N_{1,1}$.

$$\begin{aligned} b_0 N_{g,n+1}(b_0, b_S) &= \sum_{j>0} \frac{1}{2}\left[\sum_{p+q=b_0+b_j} pq N_{g,n}(b_S)|_{b_j=p} + \sum_{p+q=b_0-b_j} pq N_{g,n}(b_S)|_{b_j=p}\right] \\ &+ \sum_{p+q+r=b_0} pqr\Bigg[N_{g-1,n+2}(p, q, b_S) + \sum_{\substack{g_1+g_2=g \\ I \sqcup J = S}} N_{g_1,|I|+1}(p, b_I) N_{g_2,|J|+1}(q, b_J)\Bigg]. \end{aligned} \tag{6}$$

where $b_S=(b_1,...,b_n)$ and the sum over the term $p+q=b_0-b_j$ needs to be interpreted as follows. If $b_0-b_j>0$ it is read as written, whereas if $b_0-b_j<0$ then replace b_0-b_j by b_j-b_0 and negate the sum.

Lemma 1. *For* $\mathrm{w}_n^{(g)}(z_1,...,z_n)=\frac{d}{dz_1}...\frac{d}{dz_n}F_n^{(g)}$ *the recursion (6) is equivalent to*

$$
\begin{aligned}
\mathrm{w}_{n+1}^{(g)}(z,z_S)=&\sum_{j=1}^n\frac{\partial}{\partial z_j}\left\{\left(\frac{z^3}{(1-z^2)^2}\mathrm{w}_n^{(g)}(z_S)|_{z_j=z}-\frac{z_j^3}{(1-z_j^2)^2}\mathrm{w}_n^{(g)}(z_S)\right)\right.\\
&\qquad\left.\times\left(\frac{1}{z-z_j}+\frac{z_j}{1-zz_j}\right)\right\}\\
&+\frac{z^3}{(1-z^2)^2}\left[\mathrm{w}_{n+2}^{(g-1)}(z,z,z_S)+\sum_{\substack{g_1+g_2=g\\ I\sqcup J=S}}\mathrm{w}_{|I|+1}^{(g_1)}(z,z_I)\mathrm{w}_{|J|+1}^{(g_2)}(z,z_J)\right]
\end{aligned}\tag{7}
$$

Proof. Transform (6) by

$$\sum_{b_0=1}^\infty z^{b_0-1}\prod_{i=1}^n\sum_{b_i=1}^\infty b_iz_i^{b_i-1}\{(6)\}.$$

The left hand side transforms to $\mathrm{w}_{n+1}^{(g)}(z_0,...,z_n)$.

For each $j=0,...,n$, put $P_j=\prod_{0<i\neq j}\sum_{b_i=1}^\infty b_iz_i^{b_i-1}$. Then the first term on the right hand side of (6) transforms to

$$P_j\sum_{b_0=0}^\infty z^{b_0-1}\sum_{b_j=0}^\infty b_jz_j^{b_j-1}\sum_{p+q=b_0+b_j}\frac{1}{2}pqN_{g,n}(b_S)|_{b_j=p}$$

$$
\begin{aligned}
&=P_j\frac{\partial}{\partial z_j}\sum_{\substack{p,\,q\\ q\text{ even}}}\frac{1}{2}pqN_{g,n}(b_S)|_{b_j=p}\frac{z^{p+q}-z_j^{p+q}}{z-z_j}\\
&=P_j\frac{\partial}{\partial z_j}\sum_p pN_{g,n}(b_S)|_{b_j=p}\left[\frac{z^{p+2}}{(1-z^2)^2}-\frac{z_j^{p+2}}{(1-z_j^2)^2}\right]\frac{1}{z-z_j}\\
&=\frac{\partial}{\partial z_j}\left(\frac{z^3}{(1-z^2)^2}\mathrm{w}_n^{(g)}(z_S)|_{z_j=z}-\frac{z_j^3}{(1-z_j^2)^2}\mathrm{w}_n^{(g)}(z_S)\right)\frac{1}{z-z_j}.
\end{aligned}
$$

The transform of the second term on the right hand side of (6) breaks into two sums

$$P_j\sum_{b_0=0}^\infty z^{b_0-1}\left\{\sum_{b_j=0}^{b_0}-\sum_{b_j=b_0+1}^\infty\right\}b_jz_j^{b_j-1}\sum_{p+q=|b_j-b_0|}\frac{1}{2}pqN_{g,n}(b_S)|_{b_j=p}$$

$$= P_j \frac{\partial}{\partial z_j}\ z_j \sum_{\substack{p,\,q \\ q \text{ even}}} \frac{1}{2} pq N_{g,n}(b_S)|_{b_j=p} \frac{z^{p+q} - z_j^{p+q}}{1 - zz_j}$$

$$= P_j \frac{\partial}{\partial z_j} \sum_p p N_{g,n}(b_1, .., b_n)|_{b_j=p} \left[\frac{z^{p+2}}{(1-z^2)^2} - \frac{z_j^{p+2}}{(1-z_j^2)^2} \right] \frac{z_j}{1 - zz_j}$$

$$= \frac{\partial}{\partial z_j} \left(\frac{z^3}{(1-z^2)^2} \mathrm{w}_n^{(g)}(z_S)|_{z_j=z} - \frac{z_j^3}{(1-z_j^2)^2} \mathrm{w}_n^{(g)}(z_S) \right) \frac{z_j}{1 - zz_j}.$$

The third and fourth terms of the right hand side transform to

$$P_0 \sum_{b_0=0}^{\infty} z^{b_0-1} \sum_{p+q+r=b_0} pqr \Bigg[N_{g-1,n+2}(p, q, b_S) + \sum_{\substack{g_1+g_2=g \\ I \sqcup J = S}} N_{g_1,|I|+1}(p, b_I) N_{g_2,|J|+1}(q, b_J) \Bigg]$$

$$= P_0 \sum_{r \text{ even}} r z^{r+1} \sum_{p,q} pq \Bigg[N_{g-1,n+2}(p, q, b_S) + \sum_{\substack{g_1+g_2=g \\ I \sqcup J = S}} N_{g_1,|I|+1}(p, b_I) N_{g_2,|J|+1}(q, b_J) \Bigg] z^{p+q-2}$$

$$= P_0 \frac{z^3}{(1-z^2)^2} \sum_{p,q} pq \Bigg[N_{g-1,n+2}(p, q, b_S) + \sum_{\substack{g_1+g_2=g \\ I \sqcup J = S}} N_{g_1,|I|+1}(p, b_I) N_{g_2,|J|+1}(q, b_J) \Bigg] z^{p+q-2}$$

$$= \frac{z^3}{(1-z^2)^2} \left[\mathrm{w}_{n+2}^{(g-1)}(z, z, z_S) + \sum_{\substack{g_1+g_2=g \\ I \sqcup J = S}} \mathrm{w}_{|I|+1}^{(g_1)}(z, z_I) \mathrm{w}_{|J|+1}^{(g_2)}(z, z_J) \right].$$

Thus the Lemma is proven. □

The meromorphic form $\omega_n^{(g)}(z_1, ..., z_n)$ is defined via its Taylor expansion around $z_j = 0$, $j = 1, ..., n$ with radius of convergence 1. The following lemma gives an explicit analytic continuation of $\omega_n^{(g)}(z_1, ..., z_n)$ to $|z_j| > 1$.

Lemma 2.

$$\omega_n^{(g)}(z_1, ..., 1/z_j, ..., z_n) = -\omega_n^{(g)}(z_1, ..., z_j, ..., z_n)$$

Proof. If $p(n) = \sum_{j=0}^{k} p_j n^j$ is a polynomial then

$$\sum_{n>0} p(n) z^n = \sum_{j=0}^{k} p_j \sum_{n>0} n^j z^n = \sum_{j=0}^{k} p_j \left(z \frac{d}{dz} \right)^j \frac{z}{1-z}$$

is an expansion around $z = 0$ of a holomorphic function with radius of convergence 1 which follows from the convergence of $z + z^2 + ...$ for $|z| < 1$.

If we restrict the parity of n then

$$f_+(z) = \sum_{\substack{n>0\\ n \text{ even}}} p(n)z^n = \sum_{j=0}^{k} p_j \left(z\frac{d}{dz}\right)^j \frac{z^2}{1-z^2}$$

$$f_-(z) = \sum_{\substack{n>0\\ n \text{ odd}}} p(n)z^n = \sum_{j=0}^{k} p_j \left(z\frac{d}{dz}\right)^j \frac{z}{1-z^2}$$

are meromorphic functions with poles at $z = \pm 1$. If we further consider only polynomials in n^2, $p(n) = \sum_{j=0}^{k} p_{2j} n^{2j}$ then the extension to $|z| > 1$ is explicitly given by

$$f_+(z) + f_+(1/z) = -p(0), \quad f_-(z) + f_-(1/z) = 0$$

which follows immediately from

$$\frac{z^2}{1-z^2} + \frac{1/z^2}{1-1/z^2} = -1, \quad \frac{z}{1-z^2} + \frac{1/z}{1-1/z^2} = 0$$

and

$$\left(z\frac{d}{dz}\right)^{2j} = \left(w\frac{d}{dw}\right)^{2j}, \quad w = 1/z.$$

Hence

$$F_n^{(g)}(z_1, ..., 1/z_j, ..., z_n) = -F_n^{(g)}(z_1, ..., z_j, ..., z_n) + c(z_1, ..., \hat{z}_j, ..., z_n)$$

where $\frac{\partial}{\partial z_j} c(z_1, ..., \hat{z}_j, ..., z_n) = 0$. Thus $\omega_n^{(g)} = \frac{\partial}{\partial z_1} ... \frac{\partial}{\partial z_n} F_n^{(g)} dz_1 ... dz_n$ satisfies

$$\omega_n^{(g)}(z_1, ..., 1/z_j, ..., z_n) = -\omega_n^{(g)}(z_1, ..., z_j, ..., z_n).$$

$\square$

Proof of Theorem 1. Rewrite (7) as follows

$$\begin{aligned}\mathrm{w}_{n+1}^{(g)}(z, z_S) = \frac{z^3}{(1-z^2)^2}\Bigg\{ &\sum_{j=1}^{n} \left[\frac{1}{(z-z_j)^2} + \frac{1}{(1-zz_j)^2}\right] \mathrm{w}_n^{(g)}(z_S)|_{z_j=z} \\ &+ \mathrm{w}_{n+2}^{(g-1)}(z, z, z_S) + \sum_{\substack{g_1+g_2=g\\ I\sqcup J=S}} \mathrm{w}_{|I|+1}^{(g_1)}(z, z_I)\mathrm{w}_{|J|+1}^{(g_2)}(z, z_J)\Bigg\} \\ &- \sum_{j=1}^{n} \frac{\partial}{\partial z_j} \frac{z_j^3}{(1-z_j^2)^2} \mathrm{w}_n^{(g)}(z_S) \left(\frac{1}{z-z_j} + \frac{z_j}{1-zz_j}\right)\end{aligned} \tag{8}$$

and note that the last term is analytic at $z = \pm 1$.

In terms of $\omega_{n+1}^{(g)}(z,z_S)=\mathrm{w}_{n+1}^{(g)}(z,z_S)dzdz_S$ (8) becomes

$$\begin{aligned}\omega_{n+1}^{(g)}(z,z_S)=\frac{1}{(z-\frac{1}{z})dx(z)}\Bigg\{&\sum_{j=1}^{n}\left[\frac{dzdz_j}{(z-z_j)^2}+\frac{dzdz_j}{(1-zz_j)^2}\right]\omega_n^{(g)}(z_S)|_{z_j=z}\\&+\omega_{n+2}^{(g-1)}(z,z,z_S)+\sum_{\substack{g_1+g_2=g\\ I\sqcup J=S\\ (g_1,|I|)\neq(0,1)\\ (g_2,|J|)\neq(0,1)}}\omega_{|I|+1}^{(g_1)}(z,z_I)\omega_{|J|+1}^{(g_2)}(z,z_J)\Bigg\}\\&-\sum_{j=1}^{n}\partial_{z_j}\frac{1}{(z_j-1/z_j)dx(z_j)}\omega_n^{(g)}(z_S)\left(\frac{dz}{z-z_j}+\frac{z_jdz}{1-zz_j}\right)\end{aligned}\tag{9}$$

where various differentials have necessarily appeared on the right hand side and $\partial_{z_j}=\frac{\partial}{\partial z_j}\{\cdot\}dz_j$.

The terms

$$\frac{dzdz_j}{(z-z_j)^2}=\omega_2^{(0)}(z,z_j),\qquad\frac{dzdz_j}{(1-zz_j)^2}=-\frac{d(1/z)dz_j}{(1/z-z_j)^2}=-\omega_2^{(0)}(1/z,z_j)$$

can be absorbed into the sum over $g_1+g_2=g$ to give

$$\begin{aligned}\omega_{n+1}^{(g)}(z,z_S)=\frac{-1}{(z-\frac{1}{z})dx(z)}&\left[\omega_{n+2}^{(g-1)}(z,1/z,z_S)+\sum_{\substack{g_1+g_2=g\\ I\sqcup J=S}}\omega_{|I|+1}^{(g_1)}(z,z_I)\omega_{|J|+1}^{(g_2)}(1/z,z_J)\right]\\&-\sum_{j=1}^{n}\partial_{z_j}\frac{1}{(z_j-1/z_j)dx(z_j)}\omega_n^{(g)}(z_S)\left(\frac{dz}{z-z_j}+\frac{z_jdz}{1-zz_j}\right)\end{aligned}\tag{10}$$

and we have also used $\omega_{n'}^{(g')}(z,z_K)=-\omega_{n'}^{(g')}(1/z,z_K)$.

Apply (3) to $\omega_{n+1}^{(g)}(z_0,z_S)$ to get

$$\omega_{n+1}^{(g)}(z_0,z_S)=\sum_{\alpha=\pm1}\operatorname*{Res}_{z=\alpha}\frac{dz_0}{z_0-z}\omega_{n+1}^{(g)}(z,z_S)\tag{11}$$

We will substitute (10) into the right hand side of (11) but first note that the last term of (10) can be dropped since it is analytic at $z=\pm1$ hence does not contribute to the right hand side of (11).

$$\begin{aligned}\omega_{n+1}^{(g)}(z_0,z_S)=\sum_{\alpha=\pm1}\operatorname*{Res}_{z=\alpha}\frac{-1}{(z-\frac{1}{z})dx(z)}\frac{dz_0}{z_0-z}&\Bigg\{\omega_{n+2}^{(g-1)}(z,1/z,z_S)\\&+\sum_{\substack{g_1+g_2=g\\ I\sqcup J=S}}\omega_{|I|+1}^{(g_1)}(z,z_I)\omega_{|J|+1}^{(g_2)}(1/z,z_J)\Bigg\}\end{aligned}$$

by symmetry under $z \mapsto 1/z$

$$\omega^{(g)}_{n+1}(z_0, z_S) = \sum_{\alpha=\pm 1} \operatorname*{Res}_{z=\alpha} \frac{1}{(z-\frac{1}{z})dx(z)} \frac{dz_0}{z_0 - 1/z} \Bigg\{ \omega^{(g-1)}_{n+2}(z, 1/z, z_S) + \sum_{\substack{g_1+g_2=g \\ I \sqcup J = S}} \omega^{(g_1)}_{|I|+1}(z, z_I)\omega^{(g_2)}_{|J|+1}(1/z, z_J) \Bigg\}$$

so using

$$K(z_0, z) = \frac{-1}{2(z-\frac{1}{z})dx(z)} \left(\frac{dz_0}{z_0 - z} - \frac{dz_0}{z_0 - 1/z} \right)$$

we get

$$\omega^{(g)}_{n+1}(z_0, z_S) = \sum_{\alpha=\pm 1} \operatorname*{Res}_{z=\alpha} K(z_0, z) \Bigg[\omega^{(g-1)}_{n+2}(z, 1/z, z_S) + \sum_{\substack{g_1+g_2=g \\ I \sqcup J = S}} \omega^{(g_1)}_{|I|+1}(z, z_I)\omega^{(g_2)}_{|J|+1}(1/z, z_J) \Bigg]$$

as required. □

4. String and dilaton equations

The Eynard-Orantin invariants satisfy the following *string equations* [3].

$$\sum_{\alpha=\pm 1} \operatorname*{Res}_{z=\alpha} y(z)\omega^{(g)}_{n+1}(z, z_S) = -\sum_{j=1}^{n} dz_j \frac{\partial}{\partial z_j} \left(\frac{\omega^{(g)}_n(z_S)}{dx(z_j)} \right) \tag{12}$$

$$\sum_{\alpha=\pm 1} \operatorname*{Res}_{z=\alpha} x(z)y(z)\omega^{(g)}_{n+1}(z, z_S) = -\sum_{j=1}^{n} dz_j \frac{\partial}{\partial z_j} \left(\frac{x(z_j)\omega^{(g)}_n(z_S)}{dx(z_j)} \right). \tag{13}$$

where $z_S = (z_1, ..., z_n)$.

4.1. **String and dilaton equations for $N_{g,n}$.** The string equations (12) and (13) transform to simple equations in the $N_{g,n}$.

Proof of Theorem 2. The piecewise polynomial $N_{g,n+1}(1, b_1, ..., b_n)$ is the coefficient of $\prod_{i=i}^n b_i z_i^{b_i-1} dz_i$ in the expansion of $\operatorname*{Res}_{z=0} \frac{1}{z}\omega_{n+1}^{(g)}(z, z_S)$ around $(z, z_S) = 0$.

$$\begin{aligned}
\operatorname*{Res}_{z=0} \frac{1}{z}\omega_{n+1}^{(g)}(z, z_S) &= -\operatorname*{Res}_{z=\infty} z\omega_{n+1}^{(g)}(z, z_S) \\
&= -\operatorname*{Res}_{z=\infty} y(z)\omega_{n+1}^{(g)}(z, z_S) \\
&= \sum_{\alpha=\pm1} \operatorname*{Res}_{z=\alpha} y(z)\omega_{n+1}^{(g)}(z, z_S) \\
&= -\sum_{j=1}^n dz_j \frac{\partial}{\partial z_j}\left(\frac{\omega_n^{(g)}(z_S)}{dx(z_j)}\right) \\
&= \sum_{j=1}^n \frac{\partial}{\partial z_j}\left\{(z_j^2 + z_j^4 + z_j^6 + ...)\omega_n^{(g)}(z_S)\right\}
\end{aligned}$$

The first equality uses $\omega_{n+1}^{(g)}(z, z_S)/z = -\omega_{n+1}^{(g)}(1/z, z_S)/z$. The poles of the meromorphic form $y(z)\omega_{n+1}^{(g)}(z, z_S)$ occur at $z = -1, 1, \infty$ so the third equality uses the zero sum over all residues. The fourth equality is (12). We have expanded $1/dx(z_j) = 1/(1 - 1/z_j^2)dz_j$ around $z_j = 0$ to show that in the expansion of the final term around $z_S = 0$, the coefficient of $\prod_{i=1}^n b_i z_i^{b_i-1} dz_i$ is the right hand side of the following

$$\begin{aligned}
N_{g,n+1}(1, b_1, ..., b_n) &= \sum_{j=1}^n \sum_{\substack{k < b_j \\ k \equiv b_j + 1 (\mathrm{mod}\ 2)}} kN_{g,n}(b_1, ..., b_n)|_{b_j=k} \\
&= \sum_{j=1}^n \sum_{k=1}^{b_j} kN_{g,n}(b_1, ..., b_n)|_{b_j=k}
\end{aligned}$$

where each summand with $k \equiv b_j (\mathrm{mod}\ 2)$ vanishes since $k + \sum_{i\neq j} b_i$ is odd. This proves the first recursion of Theorem 2.

The piecewise polynomial $2N_{g,n+1}(2,b_1,...,b_n)$ appears in the expansion around $z_S=0$ of $\displaystyle\operatorname*{Res}_{z=0}\frac{1}{z^2}\omega^{(g)}_{n+1}(z,z_S)$ as the coefficient of $\prod_{i=1}^n b_iz_i^{b_i-1}dz_i$.

$$\begin{aligned}
\operatorname*{Res}_{z=0}\frac{1}{z^2}\omega^{(g)}_{n+1}(z,z_S) &= -\operatorname*{Res}_{z=\infty} z^2\omega^{(g)}_{n+1}(z,z_S)\\
&= -\operatorname*{Res}_{z=\infty} x(z)y(z)\omega^{(g)}_{n+1}(z,z_S)\\
&= \sum_{\alpha=\pm1}\operatorname*{Res}_{z=\alpha} x(z)y(z)\omega^{(g)}_{n+1}(z,z_S)\\
&= -\sum_{j=1}^n dz_j\frac{\partial}{\partial z_j}\left(\frac{x(z_j)\omega^{(g)}_n(z_S)}{dx(z_j)}\right)\\
&= \sum_{j=1}^n\frac{\partial}{\partial z_j}\left\{(z_j+2z_j^3+2z_j^5+...)\omega^{(g)}_n(z_S)\right\}
\end{aligned}$$

The first equality is as above. The second equality replaces z^2 with $z^2+1=x(z)y(z)$ since $\omega^{(g)}_{n+1}(z,z_S)$ is analytic at $z=\infty$. Again the poles of the meromorphic form $x(z)y(z)\omega^{(g)}_{n+1}(z,z_S)$ occur at $z=-1,1,\infty$ leading to the third equality. The fourth equality is (13). We have expanded $x(z_j)/dx(z_j)=(z_j-1/z_j)/(1-1/z_j^2)dz_j$ around $z_j=0$ to show that the coefficient of $\prod_{j=i}^n b_iz_i^{b_i-1}dz_i$ is the right hand side of

$$\begin{aligned}
2N_{g,n+1}(2,b_1,...,b_n) &= 2\sum_{j=1}^n\sum_{\substack{k<b_j\\ k\equiv b_j (\mathrm{mod}\ 2)}} kN_{g,n}(b_1,...,b_n)|_{b_j=k}+\sum_{j=1}^n b_jN_{g,n}(b_1,...,b_n)\\
&= 2\sum_{j=1}^n\sum_{k=1}^{b_j} kN_{g,n}(b_1,...,b_n)|_{b_j=k}-\sum_{j=1}^n b_jN_{g,n}(b_1,...,b_n)
\end{aligned}$$

where each summand with $k\equiv b_j+1(\mathrm{mod}\ 2)$ vanishes since $k+\sum_{i\neq j}b_i$ is odd. This proves the second recursion of Theorem 2. $\square$

Proof of Corollary 1. The string equations determine the genus 0 invariants. If $F(b_1^2,...,b_n^2)$ is a polynomial satisfying:

(1) degree $F(b_1^2,...,b_n^2)=n-3$
(2) $F(b_1^2,...,b_n^2)$ is symmetric in $b_1,...,b_k$
(3) $F(b_1^2,...,b_n^2)$ is symmetric in $b_{k+1},...,b_n$
(4) $F(1,b_2^2,...,b_n^2)=N^{(k)}_{g,n}(1,b_2,...,b_n)$
(5) $F(b_1^2,...,b_{n-1}^2,2^2)=N^{(k)}_{g,n}(b_1,b_2,...,b_{n-1},2)$

then $F(b_1^2,...,b_n^2)=N_{g,n}^{(k)}(b_1,b_2,...,b_n)$ since

$$\begin{aligned}F(b_1^2,...,b_n^2)-N_{g,n}^{(k)}(b_1,b_2,...,b_n)&=(b_1^2-1)(b_n^2-4)G(b_1^2,...,b_n^2)\\&=\prod_{i=1}^{k}(b_i^2-1)\prod_{j=k+1}^{n}(b_j^2-4)H(b_1^2,...,b_n^2)\end{aligned}$$

which must vanish identically to have degree $\leq n-3$. Thus $N_{g,n}^{(k)}(b_1,b_2,...,b_n)$ is uniquely determined by the two string equations. If $k=0$ or n then the argument is similar, although only one of the string equations is needed. □

The Eynard-Orantin invariants also satisfy the *dilaton equation.*

$$\sum_{\alpha=\pm1}\operatorname*{Res}_{z=\alpha}\Phi(z)\omega_{n+1}^{(g)}(z,z_S)=(2g-2+n)\omega_n^{(g)}(z_S) \tag{14}$$

where $d\Phi=ydx$. The function Φ is well-defined up to a constant in a neighbourhood of each branch point and the left hand side of (14) is independent of the choice of constant.

Proof of Theorem 3. The coefficient of $\prod_{i=1}^n b_iz_i^{b_i-1}dz_i$ in the right hand side of (14) is $(2g-2+n)N_{g,n}(b_1,...,.b_n)$. The left hand side of (14) becomes:

$$\begin{aligned}\sum_{\alpha=\pm1}\operatorname*{Res}_{z=\alpha}\Phi(z)\omega_{n+1}^{(g)}(z,z_S)&=-\sum_{\alpha=\pm1}\operatorname*{Res}_{z=\alpha}d\Phi(z)\int_0^z\omega_{n+1}^{(g)}(z',z_S)\\&=-\sum_{\alpha=\pm1}\operatorname*{Res}_{z=\alpha}(z-\frac{1}{z})dz\int_0^z\omega_{n+1}^{(g)}(z',z_S)\\&=\operatorname*{Res}_{z=\infty}(z-\frac{1}{z})dz\int_0^z\omega_{n+1}^{(g)}(z',z_S)\\&=-\operatorname*{Res}_{z=\infty}\frac{z^2}{2}\omega_{n+1}^{(g)}(z,z_S)-\operatorname*{Res}_{z=\infty}\frac{dz}{z}\int_0^z\omega_{n+1}^{(g)}(z',z_S)\\&=\operatorname*{Res}_{z=0}\frac{1}{2z^2}\omega_{n+1}^{(g)}(z,z_S)-\operatorname*{Res}_{z=\infty}\frac{dz}{z}\int_0^z\omega_{n+1}^{(g)}(z',z_S)\end{aligned}$$

The identity $0=\text{Res }d(fg)=\text{Res }df.g+\text{Res }f.dg$ applies to $f=\Phi$ and $g=\int_0^z\omega_{n+1}^{(g)}$ even though Φ is a multiply-defined function. This yields the first equality. At $z=0$ the integral $\int_0^z\omega_{n+1}^{(g)}(z',z_S)$ vanishes so $(z-1/z)\int_0^z\omega_{n+1}^{(g)}(z',z_S)$ is analytic at $z=0$ and has poles at $z=\pm1$ and ∞ which leads to the third equality. In the final expression, $N_{g,n+1}(2,b_1,...,b_n)$ is the coefficient of $\prod_{i=1}^n b_iz_i^{b_i-1}dz_i$ in the expansion of $\operatorname*{Res}_{z=0}\frac{1}{2z^2}\omega_{n+1}^{(g)}(z,z_S)$ around $(z,z_S)=0$. At $z=\infty$, $\int_0^z\omega_{n+1}^{(g)}(z',z_S)$ is analytic, thus $\operatorname*{Res}_{z=\infty}\frac{dz}{z}\int_0^z\omega_{n+1}^{(g)}(z',z_S)=\int_0^\infty\omega_{n+1}^{(g)}(z',z_S)$ which has coefficient of

$\prod_{i=1}^n b_i z_i^{b_i-1} dz_i$ given by $N_{g,n+1}(0, b_1, ..., b_n)$. Hence

$$N_{g,n+1}(2, b_1, ..., b_n) - N_{g,n+1}(0, b_1, ..., b_n) = (2g - 2 + n)N_{g,n}(b_1, ..., b_n).$$

□

Remark. It was necessary in the proofs of Theorems 2 and 3 that the $b_i > 0$. The equations immediately extend to allow all b_i. For example, the dilaton equation implies the relationship between the *polynomials*

$$N^{(k)}_{g,n+1}(2, b_1, ..., b_n) - N^{(k)}_{g,n+1}(0, b_1, ..., b_n) = (2g - 2 + n)N^{(k)}_{g,n}(b_1, ..., b_n)$$

for $b_i > 0$. The left hand side and right hand side are polynomials that agree at infinitely many values in each variable hence they coincide and in particular allow $b_i = 0$. If $b_j = 0$ in the string equation then the sum on the right hand side corresponding to j is empty. Similarly, the main recursion (6) restricts to the polynomial parts of $N_{g,n}$ and hence also allows $b_i = 0$.

4.2. **Weil-Petersson volumes of the moduli space.** Let $\mathcal{M}_{g,n}(\mathbf{L})$ be the moduli space of connected oriented genus g hyperbolic surfaces with n labeled geodesic boundary components of non-negative real lengths $L_1, ..., L_n$. It comes equipped with a symplectic form which gives rise to the Weil-Petersson volume

$$V^{WP}_{g,n}(\mathbf{L}) = \text{volume}\left[\mathcal{M}_{g,n}(\mathbf{L})\right].$$

Theorem 4 (Mirzakhani [9]). *$V^{WP}_{g,n}(\mathbf{L})$ are polynomials in $\mathbf{L} = (L_1, ..., L_n)$ that satisfy the recursion relation:*

$$\begin{aligned}\frac{\partial}{\partial L_0}(L_0 V^{WP}_{g,n+1}(L_0, \mathbf{L})) = \sum_{j=1}^{n} \int dx K_{L_0,L_j}(x) V^{WP}_{g,n}(\mathbf{L})|_{L_j=x} \\ + \int dxdy K_{L_1}(x, y)\Big\{ V^{WP}_{g-1,n+1}(x, y, \hat{\mathbf{L}}) \\ + \sum_{\substack{g_1+g_2=g \\ I \sqcup J = S}} V^{WP}_{g_1,n_1}(x, L_I) V^{WP}_{g_2,n_2}(y, L_J)\Big\}.\end{aligned}$$

As usual $S = \{1, ..., n\}$. The kernels are defined by

$$K_{L_0,L_j}(x) = H(x, L_0 + L_j) + H(x, L_0 - L_j), \quad K_{L_0}(x, y) = H(x + y, L_0)$$

for

$$H(x, y) = \frac{1}{2}\left(\frac{1}{1 + e^{\frac{x+y}{2}}} + \frac{1}{1 + e^{\frac{x-y}{2}}}\right).$$

Theorem 5 ([1]). *For $\mathbf{L} = (L_1, ..., L_n)$*

$$V^{WP}_{g,n+1}(\mathbf{L}, 2\pi i) = \sum_{k=1}^{n} \int_0^{L_k} L_k V^{WP}_{g,n}(\mathbf{L}) dL_k \tag{15}$$

$$\frac{\partial^2 V^{WP}_{g,n+1}}{\partial L^2_{n+1}}(\mathbf{L}, 2\pi i) = \mathcal{E} \cdot V^{WP}_{g,n}(\mathbf{L}) - (4g - 4 + n)V^{WP}_{g,n}(\mathbf{L}).$$

where $\mathcal{E} = \sum_{j=1}^n L_j \partial/\partial L_j$ is the Euler vector field, and

$$\frac{\partial V^{WP}_{g,n+1}}{\partial L_{n+1}}(\mathbf{L}, 2\pi i) = 2\pi i(2g - 2 + n)V^{WP}_{g,n}(\mathbf{L}). \tag{16}$$

Theorem 5 has three interesting interpretations. It was proven in [1] by showing it is equivalent to recursion relations among intersection numbers on $\overline{\mathcal{M}}_{g,n}$, the compactified moduli space of curves, that generalise the string and dilaton equations [14]. This used [10] which identified the coefficients of $V_{g,n}^{WP}$ with intersection numbers on $\overline{\mathcal{M}}_{g,n}$. Secondly, evaluation of $V_{g,n+1}^{WP}$ at $L_{n+1} = i\theta$ can be interpreted as the volume of the moduli space of hyperbolic surfaces with a cone point of angle θ. Thus Theorem 5 gives information about the moduli spaces as the cone point tends to 2π. The third view comes from the Eynard-Orantin [4] invariants. Let $\mathcal{L}\{V_{g,n}^{WP}\}(z_1,...,z_n)$ be the Laplace transform of $V_{g,n}^{WP}(L_1,...,L_n)$. Eynard and Orantin [4] proved that for $2g-2+n>0$

$$\omega_n^{(g)WP} = \frac{\partial}{\partial z_1}...\frac{\partial}{\partial z_n}\mathcal{L}\{V_{g,n}^{WP}\}dz_1...dz_n$$

are the Eynard-Orantin invariants of the plane curve $x = z^2$, $y = -\sin(2\pi z)/4\pi$ which strictly represents a sequence of algebraic curves obtained by truncating the expansion for y around $z = 0$. The first two equations in Theorem 5 correspond to the string equations and the third equation corresponds to the dilaton equation.

The string and dilaton equations satisfied by $N_{g,n}$ and $V_{g,n}^{WP}$ are strikingly similar, particularly if one substitutes $L_k = 2\pi i b_k$ and uses the analogy of discrete integration and differentiation. This suggests that the volume polynomials $V_{g,n}^{WP}$ may satisfy further identities similar to those satisfied by $N_{g,n}$ such as the vanishing results. The dilaton equation leads to the symplectic invariant $F^{(g)}$ of Eynard-Orantin and it is interesting that in both the cases of $V_{g,n}^{WP}$ and $N_{g,n}$, $F^{(g)}$ turns out to be an invariant of the classical moduli space $\mathcal{M}_g$—its volume and Euler characteristic respectively. This suggests that the general symplectic invariant $F^{(g)}$ of Eynard-Orantin is somehow related to $\mathcal{M}_g$.

4.3. **Tau notation.** Let $c_{\mathbf{m}}^{(k)}$ be the coefficient of $b_1^{2m_1}...b_n^{2m_n}$ in $N_{g,n}^{(k)}$ where the b_i have been ordered so that the first k are odd and the others are even. Define

$$\langle\tau_{m_1}^-...\tau_{m_k}^-\tau_{m_{k+1}}^+...\tau_{m_n}^+\rangle_{g,n} := 2^{2|\mathbf{m}|-g}\mathbf{m}!(3g-3+n-|\mathbf{m}|)!\times c_{\mathbf{m}}^{(k)} \tag{17}$$

where $|\mathbf{m}| = \sum_1^n m_i$ and $\mathbf{m}! = \prod_1^n m_i!$. Since $N_{g,n}^{(k)}$ is symmetric in its odd variables and its even variables this tau notation encodes the entire polynomial, and we allow the $\tau_j^{\pm}$ to be written in any order. If k is odd or $|\mathbf{m}| > 3g-3+n$ then the bracket vanishes.

The tau notation follows Witten's tau notation for intersection numbers [14]. The coefficients of the polynomials $N_{g,n}^{(k)}$ may be intersection numbers. In particular, it was proven in [11] that when $|m| = 3g-3+n$ it is an intersection number,

$$\langle\tau_{m_1}^-...\tau_{m_k}^-\tau_{m_{k+1}}^+...\tau_{m_n}^+\rangle_{g,n} = \langle\tau_{m_1}...\tau_{m_n}\rangle_{g,n} = \int_{\overline{\mathcal{M}}_{g,n}} c_1(L_1)^{m_1}...c_1(L_n)^{m_n}$$

(independent of even k) where $L_1,...,L_n$, are tautological line bundles over $\overline{\mathcal{M}}_{g,n}$.

Put $s = 3g-3+n-|\mathbf{m}|$, $\tau_{\mathbf{m}}^- = \tau_{m_1}^-...\tau_{m_k}^-$ and $\tau_{\mathbf{m}}^+ = \tau_{m_{k+1}}^+...\tau_{m_n}^+$. The string equations become

$$\sum_{p=0}^{s+1}\binom{s+1}{p}2^{-2p}\langle\tau_p^-\tau_{\mathbf{m}}^-\tau_{\mathbf{m}}^+\rangle_{g,n+1} = \sum_{p=0}^{s+1}\binom{s+1}{p}\sum_{j=1}^n b_{p,m_j}\langle\tau_{m_1}^-...\tau_{m_j+p-1}^{\mp}...\tau_{m_n}^+\rangle_{g,n}$$

$$\sum_{p=0}^{s+1}\binom{s+1}{p}\langle\tau_p^-\tau_{\mathbf{m}}^-\tau_{\mathbf{m}}^+\rangle_{g,n+1}=\sum_{p=0}^{s+1}\binom{s+1}{p}\sum_{j=1}^{n}b'_{p,m_j}\langle\tau_{m_1}^-...\tau_{m_j+p-1}^{\pm}...\tau_{m_n}^+\rangle_{g,n}$$

where b_{p,m_j} and b'_{p,m_j} depend only on p and m_j and $\tau^{\mp}_{m_j+p-1}$ reverses the parity—replace τ_j^- (respectively τ_j^+) with $\tau^+_{m_j+p-1}$ (respectively $\tau^-_{m_j+p-1}$)—and $\tau^{\pm}_{m_j+p-1}$ keeps the parity the same. When $s=-1$, both string equations reduce to the usual string equation for intersection numbers on the moduli space of curves [14]. The tau notation gives a constructive proof of Corollary 1, which states that the string equations determine the genus zero invariants, since the system of equations is triangular in the genus 0 invariants.

The dilaton equation becomes

$$\frac{1}{s+1}\sum_{m_0=1}^{s+1}\binom{s+1}{m_0}\langle\tau_{m_0}^+\tau_{\mathbf{m}}^-\tau_{\mathbf{m}}^+\rangle_{g,n+1}=(2g-2+n)\langle\tau_{\mathbf{m}}^-\tau_{\mathbf{m}}^+\rangle_{g,n}.$$

When $s=-1$, the dilaton equation reduces to the usual dilaton equation for intersection numbers on the moduli space of curves [14]. It can be used to determine the genus 1 invariants.

5. Evaluation at $b_j=0$.

The count of branched covers $N_{g,n}(b_1,...,b_n)$ requires the b_i to be positive integers since ramification 0 makes no sense. We can define $N_{g,n}(b_1,...,b_n)$ for some $b_j=0$ by evaluation of its representing polynomial $N^{(k)}_{g,n}(b_1,...,b_n)$. The dilaton equation from Theorem 3

$$N_{g,n+1}(0,b_1,...,b_n)=N_{g,n+1}(2,b_1,...,b_n)-(2g-2+n)N_{g,n}(b_1,...,b_n)$$

enables us to make sense of evaluation at $b_j=0$ in terms of a counting problem. Furthermore, as explained in the remark at the end of Section 4.1 the string and dilaton equations still hold when some $b_j=0$ and this enables us to prove vanishing results when some $b_j=0$.

Lemma 3 ([11])**.** *If* $\sum_{i=1}^{n}b_i\leq-2\chi=4g-4+2n$ *then* $N_{g,n}(b_1,...,b_n)=0$ *when all* $b_i>0$.

Proof. If $N_{g,n}(b_1,...,b_n)>0$, there exists a degree $\sum b_i$ genus g branched cover $\pi:C\to S^2$ branched over 0, 1 and ∞ with ramification $(b_1,...,b_n)$ over ∞ and ramification $(2,2,...,2)$ over 1. By the Riemann-Hurwitz formula,

$$\chi(\pi^{-1}(S^2-\{0,\infty\})=-\frac{1}{2}\sum_{i=1}^{n}b_i.$$

Thus

$$2-2g-n=\chi(\pi^{-1}(S^2-\{\infty\})=-\frac{1}{2}\sum_{i=1}^{n}b_i+\#\pi^{-1}(0)>-\frac{1}{2}\sum_{i=1}^{n}b_i.$$

□

Using the dilaton equation we can extend the vanishing result to allow some b_j to be 0.

Corollary 3. *If* $0 < \sum_{i=1}^{n} b_i \leq 4g - 4 + 2(n-p)$ *then* $N_{g,n}(b_1, ..., b_n) = 0$ *where* $\#\{b_i = 0\} = p$.

Proof. The case $p = 0$ is Lemma 3 and begins the inductive argument. Suppose $b_0 = 0$, that $\#\{b_i = 0\} = p \geq 1$ and assume

$$0 < \sum_{i=1}^{n} b_i \leq 4g - 4 + 2(n + 1 - p).$$

On the right hand side of

$$N_{g,n+1}(0, b_1, ..., b_n) = N_{g,n+1}(2, b_1, ..., b_n) - (2g - 2 + n)N_{g,n}(b_1, ..., b_n)$$

the first term vanishes by an inductive hypothesis since $\#\{b_i = 0\} = p - 1$ and

$$0 < 2 + \sum_{i=1}^{n} b_i \leq 4g - 2 + 2(n + 1 - p) = 4g - 4 + 2(n + 1 - (p - 1)).$$

The second term on the right hand side also vanishes by the inductive hypothesis since $\#\{b_i = 0\} = p - 1$ and

$$0 < \sum_{i=1}^{n} b_i \leq 4g - 4 + 2(n + 1 - p) = 4g - 4 + 2(n - (p - 1))$$

completing the induction. □

Remarks. 1. The vanishing result of Lemma 3 uses $\sum b_i \leq -2\chi$ where χ is the Euler characteristic of the cover. If we try to interpret Corollary 3 in a similar way, then we are led to the idea that the Euler characteristic should be $2 - 2g - (n - p)$ in place of $2 - 2g - n$ as if setting p of the b_i to be zero *removes p punctures.*

2. The Euler characteristic arguments of Lemma 3 and Corollary 3 cannot detect connectedness of the cover suggesting there may be further vanishing results. This is indeed the case for genus 0.

Lemma 4. *If* $0 < \sum_{i=1}^{n} b_i \leq 2(n-3)$ *then* $N_{0,n}(b_1, ..., b_n) = 0$.

Proof. This uses the main recursion relation (6) which becomes for $g = 0$

$$b_0 N_{0,n+1}(b_0, b_S) = \sum_{j>0} \frac{1}{2} \left[\sum_{p+q=b_0+b_j} pq N_{0,n}(b_S)|_{b_j=p} + \sum_{p+q=b_0-b_j} pq N_{0,n}(b_S)|_{b_j=p} \right] + \sum_{p+q+r=b_0} pqr \sum_{I \sqcup J = S} N_{0,|I|+1}(p, b_I) N_{0,|J|+1}(q, b_J)$$

where $S = \{1, ..., n\}$ and $b_S = (b_1, ..., b_n)$. The recursion allows $b_i = 0$ as explained in the remark at the end of Section 4.1.

We will prove the vanishing result by induction. If $\sum b_i$ is odd then $N_{g,n}$ vanishes so we assume $\sum b_i$ is even. When $n = 4$, if $0 < \sum b_i \leq 2$ then $(b_1, b_2, b_3, b_4) = (2, 0, 0, 0)$ or $(1, 1, 0, 0)$. These can be explicitly evaluated using

$$N_{0,4}^{(0)}(b_1, b_2, b_3, b_4) = -1 + \frac{1}{4} \sum b_i^2, \quad N_{0,4}^{(2)}(b_1, b_2, b_3, b_4) = -\frac{1}{2} + \frac{1}{4} \sum b_i^2$$

to get $N_{0,4}^{(0)}(2, 0, 0, 0) = 0 = N_{0,4}^{(2)}(1, 1, 0, 0)$ as required.

Suppose $\sum_0^n b_i \leq 2(n-2)$ and choose b_0 to be the maximum of the b_i so that we can easily interpret the second sum over $p+q=b_0-b_j$. In the first summand the variables $(b_1,...,b_j=p,...,b_n)$ satisfy

$$\sum_{i=1}^n b_i - b_j + p = \sum_{i=0}^n b_i - b_0 - b_j + p = \sum_{i=0}^n b_i - q \leq 2(n-2)-2 = 2(n-3)$$

where q must be even and $q=0$ annihilates the summand through pq so $q \geq 2$. By the inductive assumption, $N_{0,n}(b_S)|_{b_j=p}$ vanishes since the sum of its variables is less than or equal to $2(n-3)$. The variables of the second summand satisfy the same inequality by replacing the second $=$ with $\leq$ in the previous calculation since $-b_0-b_j+p \leq -b_0+b_j+p=q$ hence the second summand vanishes. The variables of the third summand satisfy

$$p+q+\sum_{i\in I} b_i + \sum_{i\in J} b_i = \sum_{i=0}^n b_i - r \leq 2(n-2)-2 = 2(n-3)$$

where as before $r \geq 2$. Hence either

$$p+\sum_{i\in I} b_i \leq 2(|I|-2) \quad \text{or} \quad q+\sum_{i\in J} b_i \leq 2(|J|-2)$$

so by the inductive assumption one of $N_{0,|I|+1}(p,b_I)$ and $N_{0,|J|+1}(q,b_J)$ vanishes so the summand vanishes and the induction is complete. □

Corollary 4.

$$\chi(\mathcal{M}_{g,n+1}) = (2-2g-n)\chi(\mathcal{M}_{g,n}).$$

Proof. By (1) $N_{g,n+1}(0,...,0) = \chi(\mathcal{M}_{g,n+1})$ so apply the dilaton equation to get

$$\chi(\mathcal{M}_{g,n+1}) = N_{g,n+1}(0,...,0) = N_{g,n+1}(2,...,0) - (2g-2+n)N_{g,n}(0,...,0).$$

Now $N_{g,n+1}(2,...,0)=0$ by Lemma 4 for $g=0$ and Corollary 3 for $g>0$ and since $N_{g,n}(0,...,0)=\chi(\mathcal{M}_{g,n})$ the result follows. □

In particular [7, 13],

$$\chi(\mathcal{M}_{g,n+1}) = (-1)^n \frac{(2g-2+n)!}{(2g-2)!}\chi(\mathcal{M}_{g,1}), \quad g>0$$
$$\chi(\mathcal{M}_{0,n+1}) = (-1)^n (n-2)!\chi(\mathcal{M}_{0,3}).$$

Furthermore, the symplectic invariant $F^{(g)}$ of Eynard and Orantin for $g>1$ is defined by applying the dilaton equation to the case $n=0$ to get

$$\chi(\mathcal{M}_{g,1}) =: (2-2g)F^{(g)}.$$

The exact sequence involving mapping class groups

$$1 \to \pi_1 C \to \Gamma_g^1 \to \Gamma_g \to 1$$

implies $\chi(\Gamma_g) = \chi(\Gamma_g^1)/\chi(C)$ and the (orbifold) Euler characteristic is $\chi(\mathcal{M}_g) = \chi(\Gamma_g)$ hence $\chi(\mathcal{M}_g) = \chi(\mathcal{M}_{g,1})/(2-2g) = F_g$ which proves Corollary 2.

Corollary 5.

$$N_{0,n}(b,0,...,0) = \prod_{k=1}^{n-3} \frac{b^2-4k^2}{4k}.$$

Proof. Lemma 4 implies that $N_{0,n}(b,0,...,0)=0$ for $b=2,...,2(n-3)$. Thus $N_{0,n}(b,0,...,0)=c\prod_{k=1}^{n-3}(b^2-4k^2)/4k$ for some constant c, since $N_{0,n}(b,0,...,0)$ is a polynomial in b^2 of degree $n-3$. Now $N_{0,n}(0,0,...,0)=\chi(\mathcal{M}_{0,n})=(-1)^{n-1}(n-3)!$ hence $c=1$. □

Using $N_{g,n}(b_1,...,b_n)$ with some $b_i=0$, one can define a compactified count of lattice points by compactifying the moduli space. It gives rise to a polynomial with constant term the Euler characteristic of the compactified moduli space.

6. Examples

In this section we give explicit formulae for the simplest Eynard-Orantin invariants $\omega_n^{(g)}$ and the corresponding piecewise polynomials $N_{g,n}$.

$$\begin{aligned}
\omega_3^{(0)} &= \left\{\frac{1}{2\prod(1-z_i)^2}-\frac{1}{2\prod(1+z_i)^2}\right\}\prod dz_i\\
\omega_1^{(1)} &= \frac{z^3dz}{(1-z^2)^4}\\
\omega_4^{(0)} &= \left\{\frac{3}{4\prod(1-z_i)^2}\sum\frac{z_i}{(1-z_i)^2}-\frac{3}{4\prod(1+z_i)^2}\sum\frac{z_i}{(1+z_i)^2}\right.\\
&\quad\left.+\frac{\sum z_iz_j(1+z_k^2)(1+z_l^2)}{2\prod(1-z_i^2)^2}\right\}\prod dz_i.\\
\omega_2^{(1)} &=\left\{\frac{5}{32\prod(1-z_i)^2}\sum\left(\frac{z_i^2}{(1-z_i)^4}-\frac{z_i}{4(1-z_i)^2}\right)+\frac{3z_0z_1}{32\prod(1-z_i)^4}\right.\\
&\qquad+\frac{5}{32\prod(1+z_i)^2}\sum\left(\frac{z_i^2}{(1+z_i)^4}+\frac{z_i}{4(1+z_i)^2}\right)+\frac{3z_0z_1}{32\prod(1+z_i)^4}\\
&\qquad\left.+\frac{z_0z_1}{8\prod(1-z_i^2)^2}\right\}dz_0dz_1\\
\omega_1^{(2)} &= \frac{21z^7(1+3z^2+z^4)dz}{(1-z^2)^{10}}
\end{aligned}$$

In $\omega_4^{(0)}$ the sum over $\{i,j,k,l\}=\{0,1,2,3\}$ consists of 24 terms.

g	**n**	# odd b_i	$N_{g,n}(b_1,...,b_n)$
0	3	0,2	1
1	1	0	$\frac{1}{48}\left(b_1^2-4\right)$
0	4	0,4	$\frac{1}{4}\left(b_1^2+b_2^2+b_3^2+b_4^2-4\right)$
0	4	2	$\frac{1}{4}\left(b_1^2+b_2^2+b_3^2+b_4^2-2\right)$
1	2	0	$\frac{1}{384}\left(b_1^2+b_2^2-4\right)\left(b_1^2+b_2^2-8\right)$
1	2	2	$\frac{1}{384}\left(b_1^2+b_2^2-2\right)\left(b_1^2+b_2^2-10\right)$
2	1	0	$\frac{1}{2^{16}3^35}\left(b_1^2-4\right)\left(b_1^2-16\right)\left(b_1^2-36\right)\left(5b_1^2-32\right)$

The piecewise polynomials are a more compact way to express the $\omega_n^{(g)}$. For example $N_{0,4}$ is the sum of five monomials whereas $\omega_4^{(0)}$ is the sum of 32 rational

functions. It may be useful to express Eynard-Orantin invariants of other curves more compactly.

Around the branch point $z = 1$ (and similarly for $z = -1$),

$$x(z) = z + 1/z = 2 + (z-1)^2 + O(z-1)^3, \quad y(z) = 1 + (z-1)$$

resembles the Airy curve $x = z^2$, $y = z$ due to the simple branching of x. Eynard and Orantin proved that near a branch point, in this case $z_i \approx 1$, $i = 1, ..., n$, the asymptotic behaviour of $\omega_n^{(g)}(z_1, ..., z_n)$ is described by $\omega_n^{(g)\mathrm{Airy}}(z_1, ..., z_n)$. The asymptotic behaviour of $\omega_n^{(g)}(z_1, ..., z_n)$ is governed by the top degree terms of the piecewise polynomial $N_{g,n}(b_1, ..., b_n)$. Since $\omega_n^{(g)\mathrm{Airy}}(z_1, ..., z_n)$ give generating functions for intersection numbers on $\overline{\mathcal{M}}_{g,n}$ this can be used to prove that the coefficients of the top degree terms of the piecewise polynomial $N_{g,n}(b_1, ..., b_n)$ are intersection numbers on $\overline{\mathcal{M}}_{g,n}$. This was proven in a different way in [11] by using the fact that the lattice point count approximates Kontsevich's volume of the moduli space which has coefficients intersection numbers on $\overline{\mathcal{M}}_{g,n}$.

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